

Oscillation and variation for Riesz transform associated with Bessel operators

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Abstract: Let $\lambda > 0$ and $\Delta_\lambda := -\frac{d^2}{dx^2} - \frac{2\lambda}{x} \frac{d}{dx}$ be the Bessel operator on $\mathbb{R}_+ := (0, \infty)$. We show that the oscillation operator $\mathcal{O}(R_{\Delta_\lambda, *})$ and variation operator $\mathcal{V}_\rho(R_{\Delta_\lambda, *})$ of the Riesz transform R_{Δ_λ} associated with Δ_λ are both bounded on $L^p(\mathbb{R}_+, dm_\lambda)$ for $p \in (1, \infty)$, from $L^1(\mathbb{R}_+, dm_\lambda)$ to $L^{1, \infty}(\mathbb{R}_+, dm_\lambda)$, and from $L^\infty(\mathbb{R}_+, dm_\lambda)$ to $BMO(\mathbb{R}_+, dm_\lambda)$, where $\rho \in (2, \infty)$ and $dm_\lambda(x) := x^{2\lambda} dx$. As an application, we give the corresponding L^p -estimates for β -jump operators and the number of up-crossing.

Keywords: oscillation; variation; Bessel operator; Riesz transform.

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1 Introduction and statement of main results

Let (\mathcal{X}, μ) be a measure space and $\mathcal{T}_* := \{T_\epsilon\}_{\epsilon>0}$ be a family of operators bounded on $L^p(\mathcal{X}, \mu)$ for $p \in (1, \infty)$ such that $\lim_{\epsilon \rightarrow 0} T_\epsilon f$ exists in some sense. The variation operator $\mathcal{V}_\rho(\mathcal{T}_*)$ and oscillation operator $\mathcal{O}(\mathcal{T}_*)$ of \mathcal{T}_* are two important tools to measure the speed of this convergence in ergodic theory; see, for example, [3, 21, 22, 23]. We recall that for any $f \in L^p(\mathbb{R}_+, dm_\lambda)$ for $p \in (1, \infty)$ and $x \in \mathcal{X}$, $\mathcal{V}_\rho(\mathcal{T}_*)(f)$ and $\mathcal{O}(\mathcal{T}_*)(f)$ are, respectively, defined by setting

$$\mathcal{V}_\rho(\mathcal{T}_*)(f)(x) := \sup_{\epsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} |T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x)|^\rho \right)^{1/\rho},$$

where the supremum is taken over all sequences $\{\epsilon_i\}$ decreasing to zero, and

$$\mathcal{O}(\mathcal{T}_*)(f)(x) := \left(\sum_{i=1}^{\infty} \sup_{\epsilon_{i+1} \leq t_{i+1} < t_i \leq \epsilon_i} |T_{t_{i+1}} f(x) - T_{t_i} f(x)|^2 \right)^{1/2}$$

with $\{\epsilon_i\}$ being a fixed sequence decreasing to zero. We also consider the operator

$$\mathcal{O}'(\mathcal{T}_*)(f)(x) = \left(\sum_{i=1}^{\infty} \sup_{\epsilon_{i+1} < \delta_i \leq \epsilon_i} |T_{\epsilon_{i+1}} f(x) - T_{\delta_i} f(x)|^2 \right)^{1/2}.$$

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It is easy to check that

$$\mathcal{O}'(\mathcal{T}_*)(f)(x) \leq \mathcal{O}(\mathcal{T}_*)(f)(x) \leq 2\mathcal{O}'(\mathcal{T}_*)(f)(x). \quad (1.1)$$

Denote by \mathcal{E} the mixed norm Banach space of two variables function h defined on $(0, \infty) \times \mathbb{N}$ such that

$$\|h\|_{\mathcal{E}} := \left(\sum_i \left(\sup_{\delta_i} |h(\delta_i, i)| \right)^2 \right)^{1/2} < \infty. \quad (1.2)$$

Then we also have that

$$\mathcal{O}'(\mathcal{T}_*)(f)(x) = \|\{T_{t_{i+1}}f(x) - T_{\delta_i}f(x)\}_{\delta_i \in (t_{i+1}, t_i], i \in \mathbb{N}}\|_{\mathcal{E}}. \quad (1.3)$$

In their remarkable work [12], Campbell *et al.* established the strong (p, p) -boundedness for $p \in (1, \infty)$ and the weak type $(1, 1)$ -boundedness of the oscillation operator and the ρ -variation operator for the Hilbert transform, and applied to the study of λ -jump operator in ergodic theory. This result was further extended in [13], to the higher dimensional cases including Riesz transforms and general singular integrals with rough homogeneous kernels in \mathbb{R}^d . Since then, boundedness of oscillation and variation operators of singular integrals operators associated with differential operators has been studied in many recent papers. In particular, Gillespie and Torrea in [19] established weighted $L^p(\mathbb{R}, \omega)$ -boundedness of the oscillation operator and the ρ -variation operator for the Hilbert transform, where $\omega \in A_p(\mathbb{R})$, the Muckenhoupt class and obtained the $L^p(\mathbb{R}^d, |x|^\alpha dx)$ -boundedness of the oscillation operator and the ρ -variation operator for Riesz transform, for $p \in (1, \infty)$ and $\alpha \in (-1, p-1)$. Later, Betancor *et al.* [8] showed that the oscillation operator and the ρ -variation operator of the Riesz transform R_{S_λ} associated with the Bessel operator S_λ for $\lambda > 0$ on $\mathbb{R}_+ := (0, \infty)$ is bounded on $L^p(\omega)$ and from $L^1(\omega)$ to $L^{1,\infty}(\omega)$, where $\omega \in A^p(\mathbb{R}_+)$ and $S_\lambda := -\frac{d^2}{dx^2} + \frac{\lambda^2 - \lambda}{x^2}$. For more results on variation and oscillation of singular integral operators, we refer the readers to [3, 5, 9, 12, 13, 16, 17, 19, 23, 24, 27, 26, 32] and the references therein.

Inspired by the result of Betancor *et al.* in [8], the aim of this paper is to prove the L^p -boundedness and their endpoint estimates of the oscillation and variation operators for Riesz transforms associated with Δ_λ , the conjugation of the Bessel operator. To this end, we recall some necessary notation.

Let λ be a positive constant. The operator Δ_λ is defined by setting, for all suitable functions f on \mathbb{R}_+ ,

$$\Delta_\lambda f(x) = -\frac{d^2}{dx^2}f(x) - \frac{2\lambda}{x} \frac{d}{dx}f(x).$$

An early work concerning the Bessel operator is from Muckenhoupt and Stein [29]. They developed a theory associated to Δ_λ which is parallel to the classical one associated to the Laplace operator Δ . After the paper [29], a lot of work concerning the Bessel operators was carried out; see, for example [1, 4, 7, 10, 11, 18, 25, 30, 31]. Among the study of Δ_λ , the properties and L^p boundedness of Riesz transforms associated to Δ_λ defined by

$$R_{\Delta_\lambda} f := \partial_x (\Delta_\lambda)^{-1/2} f,$$

($1 < p < \infty$), have been studied extensively, see for example [1, 4, 7, 29, 30]. In particular, in [7, pp.710-711], Betancor *et al.* showed that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}_+, dm_\lambda)$, then for almost every $x \in \mathbb{R}_+$,

$$R_{\Delta_\lambda} f(x) = \lim_{\varepsilon \rightarrow 0_+} R_{\Delta_\lambda, \varepsilon} f(x) := \lim_{\varepsilon \rightarrow 0_+} \int_{0, |x-y| > \varepsilon}^\infty R_{\Delta_\lambda}(x, y) f(y) dm_\lambda(y),$$

where $dm_\lambda(y) := y^{2\lambda} dy$ and for any $x, y \in \mathbb{R}_+$ with $x \neq y$,

$$R_{\Delta_\lambda}(x, y) := -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta.$$

Moreover, Betancor *et al.* in [6] characterized the atomic Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ associated to Δ_λ in terms of the Riesz transform and the radial maximal function associated with the Hankel convolution of a class of suitable functions.

Let $\rho > 2$ and $R_{\Delta_\lambda, *}, \varepsilon := \{R_{\Delta_\lambda, \varepsilon}\}_{\varepsilon > 0}$ be a family of truncated Riesz transform operators defined by

$$R_{\Delta_\lambda, \varepsilon} f(x) := \int_{0, |x-y| > \varepsilon}^\infty R_{\Delta_\lambda}(x, y) f(y) dm_\lambda(y).$$

The ρ -variation operator $\mathcal{V}_\rho(R_{\Delta_\lambda, *})$ and oscillation operator $\mathcal{O}(R_{\Delta_\lambda, *})$ associated with the Riesz transform are defined by setting, for all suitable functions f and $x \in \mathbb{R}_+$,

$$\mathcal{V}_\rho(R_{\Delta_\lambda, *}) f(x) := \sup_{\varepsilon_j \searrow 0} \left(\sum_{j=1}^\infty |R_{\Delta_\lambda, \varepsilon_j} f(x) - R_{\Delta_\lambda, \varepsilon_{j+1}} f(x)|^\rho \right)^{1/\rho}$$

with the supremum taken over all sequences $\{\varepsilon_j\}_j$ decreasing converging to zero, and

$$\mathcal{O}(R_{\Delta_\lambda, *}) f(x) := \left(\sum_{j=1}^\infty \sup_{\varepsilon_{j+1} \leq t_{j+1} < t_j \leq \varepsilon_j} |R_{\Delta_\lambda, t_j} f(x) - R_{\Delta_\lambda, t_{j+1}} f(x)|^2 \right)^{1/2},$$

where $\{\varepsilon_j\}_j$ is a fixed decreasing sequence converging to zero.

We are now to the first main result of this paper.

Theorem 1.1. *Let $\rho \in (2, \infty)$ and $\lambda > 0$. The operators $\mathcal{O}(R_{\Delta_\lambda, *})$ and $\mathcal{V}_\rho(R_{\Delta_\lambda, *})$ are bounded from $L^p(\mathbb{R}_+, dm_\lambda)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}_+, dm_\lambda)$ to $L^{1, \infty}(\mathbb{R}_+, dm_\lambda)$.*

As applications of Theorem 1.1, we consider the β -jump operators and the number of up-crossing associated with the operators sequence $\{R_{\Delta_\lambda, \varepsilon}\}_{\varepsilon > 0}$, which give certain quantitative information on the convergence of the family $\{R_{\Delta_\lambda, \varepsilon}\}_{\varepsilon > 0}$.

Definition 1.2 ([12]). Let $\beta > 0$. The β -jump operator $\Lambda(R_{\Delta_\lambda, *}, f, \beta)(x)$ associated with a sequence $R_{\Delta_\lambda, *} = \{R_{\Delta_\lambda, \varepsilon}\}_{\varepsilon > 0}$ acting on a function f at a point x is defined by

$$\Lambda(R_{\Delta_\lambda, *}, f, \beta)(x) := \sup \{n \in \mathbb{N} : \text{there exist } s_1 < t_1 \leq s_2 < t_2 < \cdots \leq s_n < t_n \\ \text{such that } |R_{\Delta_\lambda, s_i} f(x) - R_{\Delta_\lambda, t_i} f(x)| > \beta \text{ for } i = 1, 2, \dots, n\}.$$

Also, for fixed $0 < \alpha < \gamma$, we consider the number of up-crossing $N(R_{\Delta_\lambda, *}, f, \alpha, \gamma, x)$ associated with a sequence $R_{\Delta_\lambda, *} = \{R_{\Delta_\lambda, \varepsilon}\}_{\varepsilon > 0}$ acting on a function f at a point x , which is defined by

$$N(R_{\Delta_\lambda, *}, f, \alpha, \gamma, x) := \sup \left\{ n \in \mathbb{N} : \text{there exist } s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \right. \\ \left. \text{such that } R_{\Delta_\lambda, s_i} f(x) < \alpha, R_{\Delta_\lambda, t_i} f(x) > \gamma \text{ for } i = 1, 2, \dots, n \right\}.$$

It is easy to check that

$$N(R_{\Delta_\lambda,*}, f, \alpha, \gamma, x) \leq \Lambda(R_{\Delta_\lambda,*}, f, \gamma - \alpha)(x). \quad (1.4)$$

Also, from [12], the β -jump operators is controlled by the ρ -variation operator. Precisely, we have that for any $\beta \in (0, \infty)$,

$$\beta(\Lambda(R_{\Delta_\lambda,*}, f, \beta)(x))^{1/\rho} \leq \mathcal{V}_\rho(R_{\Delta_\lambda,*}f)(x). \quad (1.5)$$

As an immediate corollary of Theorem 1.1, (1.5) and (1.4), we have the following result.

Corollary 1.3. *Let $\rho \in (2, \infty)$, $\lambda, \beta \in (0, \infty)$ and $0 < \alpha < \gamma$. Then there exist positive constants $C(p, \rho, \lambda)$ and $C(\rho, \lambda)$, such that for all $f \in L^p(\mathbb{R}_+, dm_\lambda)$ with $p \in (1, \infty)$,*

$$\left\| \Lambda(R_{\Delta_\lambda,*}, f, \beta, \cdot)^{1/\rho} \right\|_{L^p(\mathbb{R}_+, dm_\lambda)} \leq \frac{C(p, \rho, \lambda)}{\beta} \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)},$$

$$\left\| N(R_{\Delta_\lambda,*}, f, \alpha, \gamma, \cdot)^{1/\rho} \right\|_{L^p(\mathbb{R}_+, dm_\lambda)} \leq \frac{C(p, \rho, \lambda)}{\gamma - \alpha} \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)},$$

and for any $f \in L^1(\mathbb{R}_+, dm_\lambda)$ and $n \geq 1$,

$$m_\lambda(\{x \in \mathbb{R}_+ : \Lambda(R_{\Delta_\lambda,*}, f, \beta, x) > n\}) \leq \frac{C(\rho, \lambda)}{\beta n^{1/\rho}} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)},$$

and

$$m_\lambda(\{x \in \mathbb{R}_+ : N(R_{\Delta_\lambda,*}, f, \alpha, \gamma, x) > n\}) \leq \frac{C(\rho, \lambda)}{(\gamma - \alpha)n^{1/\rho}} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}.$$

For $p = \infty$, we also study the boundedness of $\mathcal{O}(R_{\Delta_\lambda,*})$ and $\mathcal{V}_\rho(R_{\Delta_\lambda,*})$ from $L^\infty(\mathbb{R}_+, dm_\lambda)$ to $\text{BMO}(\mathbb{R}_+, dm_\lambda)$ introduced in [6].

Definition 1.4 ([6, 31]). A function $f \in L^1_{\text{loc}}(\mathbb{R}_+, dm_\lambda)$ belongs to the space $\text{BMO}(\mathbb{R}_+, dm_\lambda)$ if

$$\|f\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} := \sup_{x, r \in (0, \infty)} \frac{1}{m_\lambda(I(x, r))} \int_{I(x, r)} |f(y) - f_{I(x, r), \lambda}| y^{2\lambda} dy < \infty,$$

where $I(x, r) := (x - r, x + r) \cap (0, \infty)$ and

$$f_{I(x, r), \lambda} := \frac{1}{m_\lambda(I(x, r))} \int_{I(x, r)} f(y) y^{2\lambda} dy.$$

Our result concerning the boundedness of $\mathcal{O}(R_{\Delta_\lambda,*})$ and $\mathcal{V}_\rho(R_{\Delta_\lambda,*})$ for $p = \infty$ is stated as below.

Theorem 1.5. *Let $\rho \in (2, \infty)$ and $\lambda > 0$. If f in $L^\infty(\mathbb{R}_+, dm_\lambda)$ and $\mathcal{O}(R_{\Delta_\lambda,*})f(x) < \infty$ a.e. $x \in \mathbb{R}_+$, then $\mathcal{O}(R_{\Delta_\lambda,*})f \in \text{BMO}(\mathbb{R}_+, dm_\lambda)$ and there exists a positive constant C independent f such that*

$$\|\mathcal{O}(R_{\Delta_\lambda,*})f\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \leq C \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}.$$

The same result holds for $\mathcal{V}_\rho(R_{\Delta_\lambda,})$.*

The organization of this paper is as follows.

Section 2 is devoted to the proof of Theorem 1.1. We present the proof of the boundedness of $\mathcal{O}(R_{\Delta_\lambda, *})$ by dividing into two steps. In the first step, motivated by [5] and [8], we show that for any $p \in (1 + 2\lambda, \infty)$, $\mathcal{O}'(R_{\Delta_\lambda, *})$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$. Notice that though $x^{2\lambda} \in A_p(\mathbb{R}_+)$ and the kernel $R_{\Delta_\lambda}(x, y) = (xy)^\lambda R_{S_\lambda}(x, y)$, where $R_{S_\lambda}(x, y)$ is the kernel of R_{S_λ} , one can not obtain the $L^p(\mathbb{R}_+, dm_\lambda)$ -boundedness of $\mathcal{O}'(R_{\Delta_\lambda, *})$ directly by that of $\mathcal{O}'(R_{S_\lambda, *})$ in [8]. We mention that in this step, by decomposing the kernel of R_{Δ_λ} into four parts, we first have that for any $f \in L^p(\mathbb{R}_+, dm_\lambda)$ and $x \in \mathbb{R}_+$,

$$\mathcal{O}'(R_{\Delta_\lambda, *})f(x) \leq C \left[\mathcal{O}'(H_*^{loc})f(x) + T_1f(x) + T_2f(x) + \mathcal{M}_\lambda f(x) \right],$$

where H^{loc} is the local Hilbert transform introduced by Andersen and Muchenhaupt [2], \mathcal{M}_λ is the Hardy-Littlewood maximal operator on \mathbb{R}_+ with measure dm_λ and T_i , $i = 1, 2$, are bounded operators on $L^p(\mathbb{R}_+, dm_\lambda)$. Moreover, by decomposing the Hilbert transform H on \mathbb{R} into three parts, we further obtain that

$$\mathcal{O}'(H_*^{loc})f(x) \leq C \left[\mathcal{O}'(H_*)\tilde{f}(x) + \mathcal{M}f(x) + T_1f(x) \right],$$

where $\tilde{f}(x) := f(x)$ if $x \in \mathbb{R}_+$ and 0 otherwise, and \mathcal{M} is the Hardy-Littlewood maximal operator on \mathbb{R}_+ with Lebesgue measure. Then by the known fact that $x^{2\lambda} \in A_p(\mathbb{R}_+)$ if and only if $0 < 2\lambda < p - 1$ and the $L^p(\omega)$ -boundedness of $\mathcal{O}'(H_*)$, established by [19], for $p \in (1, \infty)$ and $\omega \in A_p(\mathbb{R})$, we obtain the $L^p(\mathbb{R}_+, dm_\lambda)$ -boundedness of $\mathcal{O}'(R_{\Delta_\lambda, *})$ with $p \in (1 + 2\lambda, \infty)$.

In the second step, by applying the Calderón-Zygmund decomposition established by Coifman and Weiss [14], and the $L^p(\mathbb{R}_+, dm_\lambda)$ -boundedness of $\mathcal{O}'(R_{\Delta_\lambda, *})$ with $p \in (1 + 2\lambda, \infty)$ obtained in the first step, we establish the weak type (1,1) estimation of $\mathcal{O}'(R_{\Delta_\lambda, *})$. Then by the Marcinkiewicz interpolation theorem, we further obtain the $L^p(\mathbb{R}_+, dm_\lambda)$ -boundedness of $\mathcal{O}'(R_{\Delta_\lambda, *})$ with $p \in (1, 1 + 2\lambda]$. Via (1.1), we then show that $\mathcal{O}(R_{\Delta_\lambda, *})$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$ for all $p \in (1, \infty)$ and from $L^1(\mathbb{R}_+, dm_\lambda)$ to $L^{1, \infty}(\mathbb{R}_+, dm_\lambda)$.

In Section 3, we investigate behaviors of $\mathcal{O}(R_{\Delta_\lambda, *})$ and $\mathcal{V}_\rho(R_{\Delta_\lambda, *})$ at endpoint $p = \infty$. Under the assumption that $f \in L^\infty(\mathbb{R}_+, dm_\lambda)$ with $\mathcal{O}(R_{\Delta_\lambda, *})f(x) < \infty$ a.e. x , we apply Theorem 1.1 and the known upper bound of kernel R_{Δ_λ} to establish the $(L^\infty(\mathbb{R}_+, dm_\lambda), \text{BMO}(\mathbb{R}_+, dm_\lambda))$ -boundedness of $\mathcal{O}(R_{\Delta_\lambda, *})$.

Throughout the paper, we denote by C positive constants which are independent of the main parameters, but they may vary from line to line. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we write $f \sim g$. For every $p \in (1, \infty)$, p' means the conjugate of p , i.e., $1/p' + 1/p = 1$. For any $k \in \mathbb{R}_+$ and $I := I(x, r)$ for some $x, r \in (0, \infty)$, $kI := I(x, kr)$.

2 $L^p(\mathbb{R}_+, dm_\lambda)$ -boundedness and weak type (1,1) estimate

In this section, we provide the proof of Theorem 1.1. To begin with, we first recall a useful lemma on the upper and lower bounds of kernel $R_{\Delta_\lambda}(y, z)$ of R_{Δ_λ} , which is an important tool in this paper and can be found in, for example, [7, 8, 18].

Lemma 2.1. *The kernel $R_{\Delta_\lambda}(y, z)$ satisfies the following conditions:*

- i) *There exists a positive constant C such that for any $y, z \in \mathbb{R}_+$ with $y \neq z$,*

$$|R_{\Delta_\lambda}(y, z)| \leq C \frac{1}{m_\lambda(I(y, |y - z|))}. \quad (2.1)$$

ii) There exists a positive constant \tilde{C} such that for any $y, y_0, z \in \mathbb{R}_+$ with $|y_0 - z| < |y_0 - y|/2$,

$$\begin{aligned} & |R_{\Delta_\lambda}(y, y_0) - R_{\Delta_\lambda}(y, z)| + |R_{\Delta_\lambda}(y_0, y) - R_{\Delta_\lambda}(z, y)| \\ & \leq \tilde{C} \frac{|y_0 - z|}{|y_0 - y|} \frac{1}{m_\lambda(I(y, |y_0 - y|))}. \end{aligned} \quad (2.2)$$

iii) There exist positive constants $K_1 > 2$ large enough and $C_{K_1, \lambda}, \tilde{C}_{K_1, \lambda} \in (1, \infty)$ such that for any $y, z \in \mathbb{R}_+$ with $z > K_1 y$,

$$\frac{y}{z^{2\lambda+2}} / C_{K_1, \lambda} \leq R_{\Delta_\lambda}(y, z) \leq \tilde{C}_{K_1, \lambda} \frac{y}{z^{2\lambda+2}}.$$

iv) There exist $K_2 \in (1/2, 1)$ such that $1 - K_2$ small enough and a positive constant $C_{K_2, \lambda}$ such that for any $y, z \in \mathbb{R}_+$ with $z/y \in (K_2, 1)$,

$$\left| R_{\Delta_\lambda}(y, z) + \frac{1}{\pi} \frac{1}{y^\lambda z^\lambda} \frac{1}{y - z} \right| \leq C_{K_2, \lambda} \frac{1}{y^{2\lambda+1}} \left(\log_+ \frac{\sqrt{yz}}{|y - z|} + 1 \right).$$

It is straightforward from the definition of m_λ (i.e., $dm_\lambda(x) := x^{2\lambda} dx$) that there exists a positive constant $C > 1$ such that for all $x, r \in \mathbb{R}_+$,

$$C^{-1} m_\lambda(I(x, r)) \leq x^{2\lambda} r + r^{2\lambda+1} \leq C m_\lambda(I(x, r)). \quad (2.3)$$

This means that $(\mathbb{R}_+, |\cdot|, dm_\lambda)$ is a space of homogeneous type in the sense of [14, 15].

To establish the weak type estimation for $\mathcal{O}(R_{\Delta_\lambda, *})$ and $\mathcal{V}_\rho(R_{\Delta_\lambda, *})$, another main tool in our proof is the Calderón-Zygmund decomposition established in [14, pp. 73-74] in the setting of spaces of homogeneous type.

Lemma 2.2. *Let $f \in L^1(\mathbb{R}_+, dm_\lambda)$ and $\eta > 0$, there exist functions g and b , a family of intervals $\{I_j\}_j$, and constants $C > 0$ and $M \geq 1$, such that*

- (i) $f = g + b =: g + \sum_j b_j$, where b_j is supported in I_j ,
- (ii) $\frac{1}{m_\lambda(I_j)} \int_{I_j} |b_j| dm_\lambda \leq C\eta$ and $\int_{I_j} b_j(x) dm_\lambda(x) = 0$ for each j ,
- (iii) $\|g\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} \leq C\eta$ and $\|g\|_{L^1(\mathbb{R}_+, dm_\lambda)} \leq C\|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}$,
- (iv) $\sum_j \|b_j\|_{L^1(\mathbb{R}_+, dm_\lambda)} \leq C\|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}$,
- (v) $\sum_j m_\lambda(I_j) \leq \frac{C}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}$,
- (vi) for any $x \in \mathbb{R}_+$, $\sum_j \chi_{I_j}(x) \leq M$.

Proof of Theorem 1.1. We only give the estimation $\mathcal{O}(R_{\Delta_\lambda, *})$. The proof for $\mathcal{V}_\rho(R_{\Delta_\lambda, *})$ can be given analogously as that of $\mathcal{O}(R_{\Delta_\lambda, *})$, and we leave the part to the interested readers. Moreover, by (1.1), it suffices to prove that $\mathcal{O}'(R_{\Delta_\lambda, *})$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$ for all $p \in (1, \infty)$ and from $L^1(\mathbb{R}_+, dm_\lambda)$ to $L^{1, \infty}(\mathbb{R}_+, dm_\lambda)$. We divide the proof into two steps as follows.

Step 1. We first show that $\mathcal{O}'(R_{\Delta_\lambda, *})$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$ for any $p \in (1 + 2\lambda, \infty)$. To this end, let $\{t_j\}_j$ be a fixed sequence which decreases to zero and $\delta_j \in (t_{j+1}, t_j]$ for each j , and

$$B_{\delta_j, t_{j+1}} := \{y \in \mathbb{R}_+ : t_{j+1} < |x - y| \leq \delta_j\}.$$

Motivated by the method in [5] (see also [8]), we decompose the $|R_{\Delta_\lambda, t_{j+1}} f(x) - R_{\Delta_\lambda, \delta_j} f(x)|$ according to the domain of integration as follows

$$\begin{aligned}
& |R_{\Delta_\lambda, t_{j+1}} f(x) - R_{\Delta_\lambda, \delta_j} f(x)| \\
&= \left| \int_0^{\frac{x}{2}} R_{\Delta_\lambda}(x, y) f(y) \chi_{B_{\delta_j, t_{j+1}}}(y) y^{2\lambda} dy + \int_{2x}^\infty R_{\Delta_\lambda}(x, y) f(y) \chi_{B_{\delta_j, t_{j+1}}}(y) y^{2\lambda} dy \right. \\
&\quad + \int_{\frac{x}{2}}^{2x} -\frac{1}{\pi} \frac{1}{x^\lambda y^\lambda} \frac{f(y)}{x-y} \chi_{B_{\delta_j, t_{j+1}}}(y) y^{2\lambda} dy \\
&\quad \left. + \int_{\frac{x}{2}}^{2x} \left[R_{\Delta_\lambda}(x, y) + \frac{1}{\pi} \frac{1}{x^\lambda y^\lambda} \frac{1}{x-y} \right] f(y) \chi_{B_{\delta_j, t_{j+1}}}(y) y^{2\lambda} dy \right| \\
&=: \left| \sum_{i=1}^4 \mathbf{I}_i(\delta_j, j)(x) \right|.
\end{aligned}$$

Let \mathcal{E} be the mixed norm Banach space of two variables function h defined on $(0, \infty) \times \mathbb{N}$ satisfying (1.2). Then we see that

$$\mathcal{O}'(R_{\Delta_\lambda, *})f(x) \leq \sum_{i=1}^4 \|\mathbf{I}_i(\delta_j, j)(x)\|_{\mathcal{E}}.$$

By $\|\chi_{B_{\delta_j, t_{j+1}}}(y)\|_{\mathcal{E}} \leq 1$, Lemma 2.1 i), (2.3) and Minkowski's inequality, we obtain that

$$\|\mathbf{I}_1(\delta_j, j)(x)\|_{\mathcal{E}} \leq \frac{1}{x^{2\lambda+1}} \int_0^x |f(y)| y^{2\lambda} dy \lesssim \mathcal{M}_\lambda(f)(x),$$

where \mathcal{M}_λ is the Hardy-Littlewood maximal operator defined by setting, for any function $f \in L^1_{\text{loc}}(\mathbb{R}_+, dm_\lambda)$ and $x \in \mathbb{R}_+$,

$$\mathcal{M}_\lambda(f)(x) := \sup_{\substack{I \subset \mathbb{R}_+ \\ I \ni x}} \frac{1}{m_\lambda(I)} \int_I |f(y)| y^{2\lambda} dy.$$

From [14], we know that \mathcal{M}_λ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$ with $p \in (1, \infty)$ and from $L^1(\mathbb{R}_+, dm_\lambda)$ to $L^{1, \infty}(\mathbb{R}_+, dm_\lambda)$.

Similarly, by iii) and iv) of Lemma 2.1, we have

$$\|\mathbf{I}_2(\delta_j, j)(x)\|_{\mathcal{E}} \lesssim \int_{2x}^\infty |f(y)| \frac{dy}{y} =: T_1(f)(x),$$

and

$$\|\mathbf{I}_4(\delta_j, j)(x)\|_{\mathcal{E}} \lesssim T_2(f)(x) + \frac{1}{x^{2\lambda+1}} \int_{\frac{x}{2}}^{2x} |f(y)| y^{2\lambda} dy \lesssim T_2(f)(x) + \mathcal{M}_\lambda(f)(x),$$

where

$$T_2 f(x) := \int_{\frac{x}{2}}^{2x} \frac{1}{x^{2\lambda+1}} \log_+ \frac{\sqrt{xy}}{|x-y|} |f(y)| y^{2\lambda} dy.$$

By change of variables and Minkowski's inequality, we see that for any $f \in L^p(\mathbb{R}_+, dm_\lambda)$,

$$\begin{aligned}
\|T_1(f)\|_{L^p(\mathbb{R}_+, dm_\lambda)} &= \left\{ \int_0^\infty \left[\int_2^\infty |f(xz)| \frac{dz}{z} \right]^p x^{2\lambda} dx \right\}^{1/p} \\
&\leq \int_2^\infty \left[\int_0^\infty |f(xz)|^p x^{2\lambda} dx \right]^{1/p} \frac{dz}{z} \\
&= \int_2^\infty \left[\int_0^\infty |f(y)|^p y^{2\lambda} dy \right]^{1/p} \frac{dz}{z^{(2\lambda+1)/p+1}} \\
&= \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)} \int_2^\infty z^{-(2\lambda+1)/p-1} dz \\
&= \frac{p2^{-(2\lambda+1)/p}}{2\lambda+1} \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)}. \tag{2.4}
\end{aligned}$$

On the other hand, observe that

$$\left[\int_{\frac{x}{2}}^{2x} \left(\log_+ \frac{\sqrt{xy}}{|x-y|} \right)^{p'} y^{2\lambda} dy \right]^{\frac{p}{p'}} \sim \left(\int_{\frac{1}{2}}^2 \left(\log_+ \frac{\sqrt{t}}{|1-t|} \right)^{p'} dt \right)^{\frac{p}{p'}} x^{\frac{p(2\lambda+1)}{p'}} \sim x^{\frac{p(2\lambda+1)}{p'}}.$$

By this and Hölder's inequality, we have that

$$\begin{aligned}
&\|T_2(f)\|_{L^p(\mathbb{R}_+, dm_\lambda)}^p \\
&= \int_0^\infty \frac{1}{x^{p(2\lambda+1)}} \left[\int_{\frac{x}{2}}^{2x} \log_+ \frac{\sqrt{xy}}{|x-y|} |f(y)| y^{2\lambda} dy \right]^p x^{2\lambda} dx \\
&\leq \int_0^\infty x^{2\lambda-p(2\lambda+1)} \left[\int_{\frac{x}{2}}^{2x} \left(\log_+ \frac{\sqrt{xy}}{|x-y|} \right)^{p'} y^{2\lambda} dy \right]^{\frac{p}{p'}} \left[\int_{\frac{x}{2}}^{2x} |f(y)|^p y^{2\lambda} dy \right] dx \\
&\sim \int_0^\infty x^{2\lambda-p(2\lambda+1)+\frac{p(2\lambda+1)}{p'}} \left[\int_{\frac{x}{2}}^{2x} \left(\log_+ \frac{\sqrt{xy}}{|x-y|} \right)^{p'} dy \right]^{\frac{p}{p'}} \left[\int_{\frac{x}{2}}^{2x} |f(y)|^p y^{2\lambda} dy \right] dx \\
&\sim \int_0^\infty x^{-1} \int_{\frac{x}{2}}^{2x} |f(y)|^p y^{2\lambda} dy dx \\
&\sim \int_0^\infty |f(y)|^p y^{2\lambda-1} \int_{\frac{y}{2}}^{2y} dx dy \\
&\sim \int_0^\infty |f(y)|^p y^{2\lambda} dy. \tag{2.5}
\end{aligned}$$

To estimate I_3 , let H^{loc} be the local Hilbert transform introduced by Andersen and Muckenhoupt [2]:

$$H^{\text{loc}} f(x) := p.v. - \frac{1}{\pi} \int_{\frac{x}{2}}^{2x} \frac{f(y)}{x-y} dy, \quad x > 0.$$

We write

$$H_{\delta_j, t_{j+1}}^{\text{loc}} f(x) := -\frac{1}{\pi} \int_{\frac{x}{2}}^{2x} \chi_{B_{\delta_j, t_{j+1}}}(y) \frac{f(y)}{x-y} dy, \quad x > 0.$$

Then from the mean value theorem, we deduce that

$$\begin{aligned} \left| I_3(\delta_j, j)(x) - H_{\delta_j, t_{j+1}}^{\text{loc}} f(x) \right| &= \frac{1}{\pi} \left| \int_{\frac{x}{2}}^{2x} \frac{f(y)}{x-y} \frac{y^\lambda - x^\lambda}{x^\lambda} \chi_{B_{\delta_j, t_{j+1}}}(y) dy \right| \\ &\lesssim \frac{1}{x} \int_{\frac{x}{2}}^{2x} |f(y)| \chi_{B_{\delta_j, t_{j+1}}}(y) dy. \end{aligned}$$

This implies that

$$\|I_3(\delta_j, j)(x)\|_{\mathcal{E}} \lesssim \|H_{\delta_j, t_{j+1}}^{\text{loc}} f\|_{\mathcal{E}} + \mathcal{M}_\lambda f(x).$$

Observe that

$$H^{\text{loc}} f(x) = H\tilde{f}(x) + \frac{1}{\pi} \int_0^{x/2} \frac{f(y)}{x-y} dy + \frac{1}{\pi} \int_{2x}^\infty \frac{f(y)}{x-y} dy,$$

where H is the Hilbert transform and

$$\tilde{f}(x) = \begin{cases} f(x), & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} H_{\delta_j, t_{j+1}}^{\text{loc}} f(x) &= H_{\delta_j, t_{j+1}} \tilde{f}(x) + \frac{1}{\pi} \int_0^{x/2} \chi_{B_{\delta_j, t_{j+1}}}(y) \frac{f(y)}{x-y} dy \\ &\quad + \frac{1}{\pi} \int_{2x}^\infty \chi_{B_{\delta_j, t_{j+1}}}(y) \frac{f(y)}{x-y} dy. \end{aligned}$$

Then by (1.3) we see that

$$\begin{aligned} \mathcal{O}'(H_*^{\text{loc}})f(x) &= \|H_{\delta_j, t_{j+1}}^{\text{loc}} f\|_{\mathcal{E}} \leq \|H_{\delta_j, t_{j+1}} \tilde{f}\|_{\mathcal{E}} + \frac{1}{\pi} \int_0^{x/2} \frac{|f(y)|}{|x-y|} dy + \frac{1}{\pi} \int_{2x}^\infty \frac{|f(y)|}{|x-y|} dy \\ &\leq \mathcal{O}'(H_*)\tilde{f}(x) + \frac{2}{x\pi} \int_0^x |f(y)| dy + \frac{2}{\pi} \int_{2x}^\infty \frac{|f(y)|}{y} dy \\ &\lesssim \mathcal{O}'(H_*)\tilde{f}(x) + \mathcal{M}f(x) + T_1 f(x), \end{aligned}$$

where $\mathcal{M}f(x)$ is the Hardy-Littlewood maximal function defined as, for any $f \in L_{\text{loc}}^1(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$,

$$\mathcal{M}f(x) := \sup_{\substack{I \subset \mathbb{R}_+ \\ I \ni x}} \frac{1}{|I|} \int_I |f(y)| dy.$$

Consequently, we have

$$\mathcal{O}'(R_{\Delta_\lambda, *})f(x) \lesssim \mathcal{O}'(H_*)\tilde{f}(x) + \mathcal{M}f(x) + T_1 f(x) + T_2 f(x) + \mathcal{M}_\lambda f(x).$$

From [19, Theorem 1.5], we have $\mathcal{O}'(H_*) : L^p(\omega) \rightarrow L^p(\omega)$, $\omega \in A_p(\mathbb{R})$. And from [28], we also have \mathcal{M} is bounded on $L^p(\omega)$. Moreover, from [20, p. 286], we know that $x^{2\lambda} \in A_p$ if and only if $0 < 2\lambda < p - 1$. Thus combining (2.4) and (2.5), we get $\mathcal{O}'(R_{\Delta_\lambda, *})f(x)$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$ for $p \in (1 + 2\lambda, \infty)$.

Step 2. By applying the Marcinkiewicz interpolation theorem and Step 1, to finish the proof of Theorem 1.1, it suffices to prove that for any $f \in L^1(\mathbb{R}_+, dm_\lambda)$ and $\eta > 0$,

$$m_\lambda(\{x \in \mathbb{R}_+ : \mathcal{O}'(R_{\Delta_\lambda, *})f(x) > \eta\}) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}. \quad (2.6)$$

The main tool in our proof is the Calderón-Zygmund decomposition in Lemma 2.2. Let $g, b, \{b_j\}_j$ and $\{I_j\}_j$ be as Lemma 2.2. Since the operator $\mathcal{O}'(R_{\Delta_\lambda, *})$ is sublinear, to show (2.6), it suffices to prove

$$m_\lambda\left(\left\{x \in \mathbb{R}_+ : |\mathcal{O}'(R_{\Delta_\lambda, *})(g)(x)| > \frac{\eta}{2}\right\}\right) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}, \quad (2.7)$$

and

$$m_\lambda\left(\left\{x \in \mathbb{R}_+ : |\mathcal{O}'(R_{\Delta_\lambda, *})(b)(x)| > \frac{\eta}{2}\right\}\right) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}. \quad (2.8)$$

For (2.7), by the L^p -boundedness of $\mathcal{O}'(R_{\Delta_\lambda, *})$ with $p > 1 + 2\lambda$ from Step 1 and Lemma 2.2 (iii), we have

$$\begin{aligned} m_\lambda\left(\left\{x \in \mathbb{R}_+ : |\mathcal{O}'(R_{\Delta_\lambda, *})(g)(x)| > \frac{\eta}{2}\right\}\right) &\lesssim \frac{1}{\eta^p} \int_{\mathbb{R}_+} [\mathcal{O}'(R_{\Delta_\lambda, *})(g)(x)]^p dm_\lambda(x) \\ &\lesssim \frac{1}{\eta^p} \int_{\mathbb{R}_+} |g(x)|^p dm_\lambda(x) \\ &\lesssim \frac{1}{\eta} \int_{\mathbb{R}_+} |f(x)| dm_\lambda(x). \end{aligned}$$

This shows (2.7).

In what follows, we prove (2.8). Let $\tilde{I}_j := 3I_j$ and $\tilde{\mathcal{I}} := \bigcup_j \tilde{I}_j$. Using the doubling property of m_λ (2.3) and Lemma 2.2 (v), we write

$$\begin{aligned} m_\lambda\left(\left\{x \in \mathbb{R}_+ : |\mathcal{O}'(R_{\Delta_\lambda, *})(b)(x)| > \frac{\eta}{2}\right\}\right) &\lesssim m_\lambda(\tilde{\mathcal{I}}) + m_\lambda\left(\left\{x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : |\mathcal{O}'(R_{\Delta_\lambda, *})(b)(x)| > \frac{\eta}{2}\right\}\right) \\ &\lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)} + m_\lambda\left(\left\{x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : |\mathcal{O}'(R_{\Delta_\lambda, *})(b)(x)| > \frac{\eta}{2}\right\}\right). \end{aligned}$$

It remains to estimate the second term on the right of the last inequality. To this end, we first introduce some notations. Let $\delta_i \in (t_{i+1}, t_i]$ and A_{δ_i} be the interval $(t_{i+1}, \delta_i]$. Set

$$R_{\Delta_\lambda}^{A_{\delta_i}} b(x) := \int_{|x-y| \in A_{\delta_i}} R_{\Delta_\lambda}(x, y) b(y) dm_\lambda(y).$$

Then, the operator $\mathcal{O}'(R_{\Delta_\lambda, *})(b)(x)$ can be expressed more conveniently as:

$$\mathcal{O}'(R_{\Delta_\lambda, *})(b)(x) = \left[\sum_{i=1}^{\infty} \sup_{\delta_i \in (t_{i+1}, t_i]} \left| R_{\Delta_\lambda}^{A_{\delta_i}}(b)(x) \right|^2 \right]^{1/2}.$$

For every $x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}}$, choose a $\tilde{\delta}_i \in (t_{i+1}, t_i]$ such that

$$\mathcal{O}'(R_{\Delta_\lambda, *})(b)(x) \leq \left[\sum_i 2 \left| R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b)(x) \right|^2 \right]^{1/2} \leq 2 \left[\sum_i \left| \sum_j R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2}.$$

Then we only need to prove the following inequality:

$$m_\lambda\left(\left\{x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \left(\sum_i \left|\sum_j R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x)\right|^2\right)^{1/2} > \frac{\eta}{4}\right\}\right) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}. \quad (2.9)$$

For each $x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}}$ and i ,

$$A_{\tilde{\delta}_i} + x := (x + t_{i+1}, x + \tilde{\delta}_i] \cup ([x - \tilde{\delta}_i, x - t_{i+1}) \cap \mathbb{R}_+).$$

Note that $R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x)$ is nonzero only if, for some j , I_j lies entirely in $A_{\tilde{\delta}_i} + x$ or I_j intersects the boundary of $A_{\tilde{\delta}_i} + x$. So, we split the family of indexes j in two categories:

$$L_i^1 := \{j : I_j \subset x + A_{\tilde{\delta}_i}\} \quad \text{and} \quad L_i^2 := \{j : I_j \not\subset x + A_{\tilde{\delta}_i} \text{ and } I_j \cap (x + A_{\tilde{\delta}_i}) \neq \emptyset\}.$$

Thus,

$$\begin{aligned} & \left[\sum_i \left| \sum_j R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2} \\ & \leq \left[\sum_i \left| \sum_{j \in L_i^1} R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2} + \left[\sum_i \left| \sum_{j \in L_i^2} R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2}. \end{aligned}$$

Hence, to prove (2.9), it suffices to show the following two inequalities:

$$m_\lambda\left(\left\{x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \left[\sum_i \left| \sum_{j \in L_i^1} R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2} > \frac{\eta}{8}\right\}\right) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}, \quad (2.10)$$

and

$$m_\lambda\left(\left\{x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \left[\sum_i \left| \sum_{j \in L_i^2} R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2} > \frac{\eta}{8}\right\}\right) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}.$$

We first prove (2.10). Fix $i \in \mathbb{N}$ and let $j \in L_i^1$, that is, $I_j \subset x + A_{\tilde{\delta}_i}$. Since by Lemma 2.2 (ii), $\int_{\mathbb{R}_+} b_j(x) dm_\lambda(x) = 0$, we have

$$R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) = \int_{\mathbb{R}_+} R_{\Delta_\lambda}(x, y) b_j(y) dm_\lambda(y) = \int_{\mathbb{R}_+} [R_{\Delta_\lambda}(x, y) - R_{\Delta_\lambda}(x, y_j)] b_j(y) dm_\lambda(y), \quad (2.11)$$

where y_j is the center of I_j . Because for fixed x , $x + A_{\tilde{\delta}_i}$ and $x + A_{\tilde{\delta}_k}$ are disjoint for $i \neq k$, we see that for $j \in L_i^1$, I_j and $x + A_{\tilde{\delta}_k}$ are disjoint. Therefore, by (2.2) and (2.11),

$$\begin{aligned} \left[\sum_i \left| \sum_{j \in L_i^1} R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2} & \leq \sum_i \left| \sum_{j \in L_i^1} R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right| \\ & \lesssim \sum_j \int_{\mathbb{R}_+} |R_{\Delta_\lambda}(x, y) - R_{\Delta_\lambda}(x, y_j)| |b_j(y)| dm_\lambda(y) \\ & \lesssim \sum_j \int_{\mathbb{R}_+} \frac{|y - y_j|}{|x - y|} \frac{1}{m_\lambda(I(x, |x - y|))} |b_j(y)| dm_\lambda(y) \end{aligned}$$

Thus we have

$$\begin{aligned}
& m_\lambda \left(\left\{ x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \left[\sum_i \left| \sum_{j \in L_i^1} R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right|^2 \right]^{1/2} > \frac{\eta}{8} \right\} \right) \\
& \lesssim \frac{1}{\eta} \int_{\mathbb{R}_+ \setminus \tilde{\mathcal{I}}} \sum_j \int_{\mathbb{R}_+} \frac{|y - y_j|}{|x - y|} \frac{1}{m_\lambda(I(x, |x - y|))} |b_j(y)| dm_\lambda(y) dm_\lambda(x) \\
& \lesssim \frac{1}{\eta} \sum_j \int_{I_j} |b_j(y)| \int_{\mathbb{R}_+ \setminus \tilde{\mathcal{I}}} \frac{|y - y_j|}{|x - y|} \frac{1}{m_\lambda(I(x, |x - y|))} dm_\lambda(x) dm_\lambda(y) \\
& \lesssim \frac{1}{\eta} \sum_j \int_{I_j} |b_j(y)| \sum_{k=1}^{\infty} \frac{|I_j|}{3^k |I_j|} \int_{3^{k+1} I_j \setminus 3^k I_j} \frac{dm_\lambda(x)}{m_\lambda(I(x, |x - y|))} dm_\lambda(y).
\end{aligned}$$

Note that for any $x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}}$, $y, y_j \in I_j$, we have $|x - y| \sim |x - y_j|$ and

$$m_\lambda(I(x, |x - y|)) \sim m_\lambda(I(x, |x - y_j|)) \sim m_\lambda(I(y_j, |x - y_j|)). \quad (2.12)$$

Thus, we get

$$\int_{3^{k+1} I_j \setminus 3^k I_j} \frac{dm_\lambda(x)}{m_\lambda(I(x, |x - y|))} \lesssim \frac{m_\lambda(3^{k+1} I_j)}{m_\lambda(3^k I_j)} \lesssim 1.$$

Consequently, by this fact and Lemma 2.2 (iv), we conclude that

$$m_\lambda \left(\left\{ x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \left(\sum_i \left| \sum_{j \in L_i^1} R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right| \right) > \frac{\eta}{8} \right\} \right) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)}.$$

This completes the proof of (2.10).

For the part L_i^2 , a simple geometrical inspection via Lemma 2.2(vi) shows that L_i^2 contains at most finite j 's for any i . It then follows that

$$\begin{aligned}
\sum_i \left| \sum_{j \in L_i^2} R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right|^2 & \lesssim \sum_i \sum_{j \in L_i^2} \left| R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right|^2 \\
& \lesssim \sum_j \sum_i \left| R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right|^2 \\
& \lesssim \sum_j \left(\sum_i \left| R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right| \right)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
& m_\lambda \left(\left\{ x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \left(\sum_i \left| \sum_{j \in L_i^2} R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right|^2 \right)^{1/2} > \frac{\eta}{8} \right\} \right) \\
& \leq m_\lambda \left(\left\{ x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \sum_i \left| \sum_{j \in L_i^2} R_{\Delta_\lambda}^{A_{\delta_i}}(b_j)(x) \right|^2 > \frac{\eta^2}{64} \right\} \right)
\end{aligned}$$

$$\lesssim \frac{1}{\eta^2} \sum_j \int_{\mathbb{R}_+ \setminus \tilde{I}_j} \left(\sum_i \left| R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right| \right)^2 dm_\lambda(x).$$

Let $x \in \mathbb{R}_+ \setminus \tilde{I}_j$ for some j . Then by (2.1) and (2.12), we have

$$\begin{aligned} \sum_i |R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x)|^2 &\leq \sum_i \left[\int_{|x-y| \in A_{\tilde{\delta}_i}} |R_{\Delta_\lambda}(x, y)| |b_j(y)| dm_\lambda(y) \right]^2 \\ &\lesssim \sum_i \left[\int_{|x-y| \in A_{\tilde{\delta}_i}} \frac{|b_j(y)|}{m_\lambda(I(x, |x-y|))} dm_\lambda(y) \right]^2 \\ &\lesssim \frac{1}{[m_\lambda(I(y_j, |x-y_j|))]^2} \int_{\mathbb{R}_+} |b_j(y)| dm_\lambda(y) \\ &\quad \times \sum_i \int_{|x-y| \in A_{\tilde{\delta}_i}} \frac{|b_j(y)|}{m_\lambda(I(x, |x-y|))} dm_\lambda(y) \\ &\lesssim \frac{1}{[m_\lambda(I(y_j, |x-y_j|))]^2} \left[\int_{\mathbb{R}_+} |b_j(y)| dm_\lambda(y) \right]^2. \end{aligned}$$

Therefore, by (ii) and (iv) of Lemma 2.2, we get

$$\begin{aligned} &\int_{\mathbb{R}_+ \setminus \tilde{I}_j} \left(\sum_i \left| R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right| \right)^2 dm_\lambda(x) \\ &\lesssim \int_{\mathbb{R}_+ \setminus \tilde{I}_j} \frac{1}{[m_\lambda(I(y_j, |x-y_j|))]^2} \left(\int_{\mathbb{R}_+} |b_j(y)| dm_\lambda(y) \right)^2 dm_\lambda(x) \\ &\lesssim \sum_{k=1}^{\infty} \int_{3^{k+1}I_j \setminus 3^k I_j} \frac{\eta m_\lambda(I_j)}{[m_\lambda(I(y_j, |x-y_j|))]^2} dm_\lambda(x) \int_{\mathbb{R}_+} |b_j(y)| dm_\lambda(y) \\ &\lesssim \eta \int_{\mathbb{R}_+} |b_j(y)| dm_\lambda(y) \sum_{k=1}^{\infty} \frac{m_\lambda(I_j) m_\lambda(3^{k+1}I_j)}{[m_\lambda(3^k I_j)]^2} \\ &\lesssim \eta \int_{\mathbb{R}_+} |b_j(y)| dm_\lambda(y). \end{aligned}$$

Consequently, we get

$$m_\lambda \left(\left\{ x \in \mathbb{R}_+ \setminus \tilde{\mathcal{I}} : \left[\sum_j \left| \sum_{i \in L_i^2} R_{\Delta_\lambda}^{A_{\tilde{\delta}_i}}(b_j)(x) \right|^2 \right]^{1/2} > \frac{\eta}{4} \right\} \right) \lesssim \frac{1}{\eta} \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)},$$

which finishes the proof of weak type $(1, 1)$ estimation of $\mathcal{O}'(R_{\Delta_\lambda, *})$. Theorem 1.1 is proved. \square

3 Proof of Theorem 1.5

As the proof of Theorem 1.1, by similarity, we only consider $\mathcal{O}(R_{\Delta_\lambda, *})$. The proof for $\mathcal{V}_\rho(R_{\Delta_\lambda, *})$ is similar and omitted.

Fix $f \in L^\infty(\mathbb{R}_+, dm_\lambda)$ and define $f_1(y) := f(y)\chi_{4I}$ and $f_2(y) := f(y)\chi_{\mathbb{R}_+ \setminus 4I}$, where $I := I(x_0, r)$. From the Hölder inequality, Theorem 1.1 and (2.3), we deduce that

$$\int_I |\mathcal{O}(R_{\Delta_\lambda, *})(f_1)(x)| dm_\lambda(x) \leq [m_\lambda(I)]^{\frac{1}{2}} \left[\int_{\mathbb{R}_+} |\mathcal{O}(R_{\Delta_\lambda, *})(f_1)(x)|^2 dm_\lambda(x) \right]^{\frac{1}{2}}$$

$$\begin{aligned}
&\lesssim [m_\lambda(I)]^{\frac{1}{2}} \left[\int_{4I} |f(x)|^2 dm_\lambda(x) \right]^{\frac{1}{2}} \\
&\lesssim m_\lambda(I) \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}.
\end{aligned} \tag{3.1}$$

Then $\mathcal{O}(R_{\Delta_\lambda, *})(f_1)(x) < \infty$, *a.e.* $x \in I$. According to the assumption, we may choose $x_1 \in I$ such that $\mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x_1) < \infty$.

By the fact that $\mathcal{O}(R_{\Delta_\lambda, *})$ is sublinear, (3.1), we write

$$\begin{aligned}
&\frac{1}{m_\lambda(I)} \int_I |\mathcal{O}(R_{\Delta_\lambda, *})(f)(x) - \mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x_1)| dm_\lambda(x) \\
&\leq \frac{1}{m_\lambda(I)} \int_I \mathcal{O}(R_{\Delta_\lambda, *})(f_1)(x) dm_\lambda(x) \\
&\quad + \frac{1}{m_\lambda(I)} \int_I |\mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x) - \mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x_1)| dm_\lambda(x) \\
&\lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} + \frac{1}{m_\lambda(I)} \int_I |\mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x) - \mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x_1)| dm_\lambda(x).
\end{aligned}$$

Thus, to finish the proof of Theorem 1.5, it suffices to show that

$$\frac{1}{m_\lambda(I)} \int_I |\mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x) - \mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x_1)| dm_\lambda(x) \lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}.$$

Assume that $\mathcal{O}(R_{\Delta_\lambda, *})$ is defined by a given sequence $\{t_i\}_i$ decreasing and converging to zero. Now, for $x \in I$, and $I_i := (t_{i+1}, t_i]$, $i \in \mathbb{N}$, by (1.1) and (1.3), we write

$$\begin{aligned}
&|\mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x) - \mathcal{O}(R_{\Delta_\lambda, *})(f_2)(x_1)| \\
&\leq \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_{i+1} < \epsilon_i \leq t_i} \left| [R_{\Delta_\lambda, \epsilon_i} f_2(x) - R_{\Delta_\lambda, \epsilon_{i+1}} f_2(x)] \right. \right. \\
&\quad \left. \left. - [R_{\Delta_\lambda, \epsilon_i} f_2(x_1) - R_{\Delta_\lambda, \epsilon_{i+1}} f_2(x_1)] \right|^2 \right)^{1/2} \\
&\lesssim \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} < \delta_i \leq t_i} \left| [R_{\Delta_\lambda, t_{i+1}} f_2(x) - R_{\Delta_\lambda, \delta_i} f_2(x)] \right. \right. \\
&\quad \left. \left. - [R_{\Delta_\lambda, t_{i+1}} f_2(x_1) - R_{\Delta_\lambda, \delta_i} f_2(x_1)] \right|^2 \right)^{1/2} \\
&= \left\| \left\{ \int_{\{t_{i+1} < |x-y| \leq \delta_i\}} R_{\Delta_\lambda}(x, y) f_2(y) dm_\lambda(y) \right. \right. \\
&\quad \left. \left. - \int_{\{t_{i+1} < |x_1-y| \leq \delta_i\}} R_{\Delta_\lambda}(x_1, y) f_2(y) dm_\lambda(y) \right\}_{\delta_i \in I_i, i \in \mathbb{N}} \right\|_{\mathcal{E}} \\
&\leq \left\| \left\{ \int_{\{t_{i+1} < |x-y| \leq \delta_i\}} [R_{\Delta_\lambda}(x, y) - R_{\Delta_\lambda}(x_1, y)] f_2(y) dm_\lambda(y) \right\}_{\delta_i \in I_i, i \in \mathbb{N}} \right\|_{\mathcal{E}} \\
&\quad + \left\| \left\{ \int_{\mathbb{R}_+} [\chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) - \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y)] R_{\Delta_\lambda}(x_1, y) f_2(y) dm_\lambda(y) \right\}_{\delta_i \in I_i, i \in \mathbb{N}} \right\|_{\mathcal{E}} \\
&=: D_1 + D_2.
\end{aligned}$$

Notice that

$$\|\{\chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y)\}_{\delta_i \in I_i, i \in \mathbb{N}}\|_{\mathcal{E}} \leq 1.$$

By this fact, Mikowski's inequality, (2.2) and (2.3), we have

$$\begin{aligned}
D_1 &\leq \int_{\mathbb{R}_+} \|\{\chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y)\}_{\delta_i \in J_i, i \in \mathbb{N}}\| \varepsilon |R_{\Delta_\lambda}(x, y) - R_{\Delta_\lambda}(x_1, y)| |f_2(y)| dm_\lambda(y) \\
&\leq \int_{\mathbb{R}_+} |R_{\Delta_\lambda}(x, y) - R_{\Delta_\lambda}(x_1, y)| |f_2(y)| dm_\lambda(y) \\
&\lesssim \sum_{k=2}^{\infty} \int_{2^{k+1}I \setminus 2^k I} \frac{|x - x_1| |f(y)|}{|x - y| m_\lambda(I(y, |x - y|))} dm_\lambda(y) \\
&\lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} \sum_{k=2}^{\infty} \frac{|I|}{2^k |I|} \int_{2^{k+1}I \setminus 2^k I} \frac{dm_\lambda(y)}{m_\lambda(I(y, |x - y|))} \\
&\lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} \sum_{k=2}^{\infty} \frac{1}{2^k} \frac{m_\lambda(2^{k+1}I)}{m_\lambda(2^k I)} \\
&\lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)},
\end{aligned}$$

where in the third-to-last inequality, we use the fact that for $x, x_0 \in I$, $y \in \mathbb{R}_+ \setminus 4I$, $|x - y| \sim |x_0 - y|$, and in the second-to-last inequality,

$$m_\lambda(I(y, |x - y|)) \sim m_\lambda(I(y, |x_0 - y|)) \sim m_\lambda(I(x_0, |x_0 - y|)). \quad (3.2)$$

For D_2 , note that the integral

$$E := \int_{\mathbb{R}_+} |\chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) - \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y)| |R_{\Delta_\lambda}(x_1, y)| |f_2(y)| dm_\lambda(y) \neq 0$$

only if either

$$\chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) = 1 \quad \text{and} \quad \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) = 0,$$

or viceversa, that is,

$$\chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) = 0 \quad \text{and} \quad \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) = 1.$$

Equivalently, if $E \neq 0$, at least one of the following four statements holds:

- (i) $t_{i+1} < |x - y| \leq \delta_i$ and $|x_1 - y| \leq t_{i+1}$;
- (ii) $t_{i+1} < |x - y| \leq \delta_i$ and $|x_1 - y| > \delta_i$;
- (iii) $|x - y| \leq t_{i+1}$ and $t_{i+1} < |x_1 - y| \leq \delta_i$;
- (iv) $|x - y| > \delta_i$ and $t_{i+1} < |x_1 - y| \leq \delta_i$.

Since $|x - x_1| < 2r$, we observe that in case (i),

$$t_{i+1} < |x - y| \leq |x - x_1| + |x_1 - y| < t_{i+1} + 2r;$$

in case (ii),

$$\delta_i < |x_1 - y| \leq |x_1 - x| + |x - y| < \delta_i + 2r;$$

in case (iii),

$$t_{i+1} < |x_1 - y| < t_{i+1} + 2r;$$

and in case (iv),

$$\delta_i < |x - y| < \delta_i + 2r.$$

Then, we write

$$\begin{aligned} E &\lesssim \int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \chi_{\{t_{i+1} < |x-y| < t_{i+1}+2r\}}(y) |R_{\Delta_\lambda}(x_1, y)| |f_2(y)| dm_\lambda(y) \\ &\quad + \int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \chi_{\{\delta_i < |x_1-y| < \delta_i+2r\}}(y) |R_{\Delta_\lambda}(x_1, y)| |f_2(y)| dm_\lambda(y) \\ &\quad + \int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) \chi_{\{t_{i+1} < |x_1-y| < t_{i+1}+2r\}}(y) |R_{\Delta_\lambda}(x_1, y)| |f_2(y)| dm_\lambda(y) \\ &\quad + \int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) \chi_{\{\delta_i < |x-y| < \delta_i+2r\}}(y) |R_{\Delta_\lambda}(x_1, y)| |f_2(y)| dm_\lambda(y) \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We now estimate J_1 and claim that

$$J_1 \lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{r}{|x_0 - y|} \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0 - y|))} dm_\lambda(y) \right]^{1/2}. \quad (3.3)$$

Indeed, observe that for any $y \in \mathbb{R}_+ \setminus 4I$ and $x \in I$,

$$|x - y| > 3r, \quad |x_1 - y| > 3r, \quad \text{and} \quad |x_1 - y|/3 < |x - y| < 5|x_1 - y|/3. \quad (3.4)$$

Moreover, if $x, x_1 \in I$ and $r \geq t_{i+1}$, $i \in \mathbb{N}$, we have

$$\{y \in \mathbb{R}_+ \setminus 4I : t_{i+1} < |x - y| < t_{i+1} + 2r\} \subset \{y \in \mathbb{R}_+ \setminus 4I : |x - y| < 3r\} = \emptyset.$$

This means $J_1 = 0$ and (3.3) holds.

In the following, we assume that $r < t_{i+1}$. From this assumption, we further deduce that for $x \in I$,

$$0 < m_\lambda(I(x, t_{i+1} + 2r)) - m_\lambda(I(x, t_{i+1})) \lesssim (x + t_{i+1} + 2r)^{2\lambda} r \lesssim (x + t_{i+1})^{2\lambda} r.$$

By this fact, together with (2.1), (2.3), Hölder's inequality and the fact that for any $x \in I$, $y \in \mathbb{R}_+ \setminus 4I$,

$$m_\lambda(I(x_1, |x_1 - y|)) \sim m_\lambda(I(y, |x_1 - y|)) \sim m_\lambda(I(y, |x - y|)) \sim m_\lambda(I(x, |x - y|)), \quad (3.5)$$

we see that

$$\begin{aligned} J_1 &\leq [m_\lambda(I(x, t_{i+1} + 2r)) - m_\lambda(I(x, t_{i+1}))]^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) |R_{\Delta_\lambda}(x_1, y)|^2 |f_2(y)|^2 dm_\lambda(y) \right)^{1/2} \\ &\lesssim \left[(x + t_{i+1})^{2\lambda} r \int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_1, |x_1 - y|))^2} dm_\lambda(y) \right]^{1/2} \\ &\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_1, |x_1 - y|))} \frac{(x + t_{i+1})^{2\lambda} r}{m_\lambda(I(x, |x - y|))} dm_\lambda(y) \right]^{1/2} \end{aligned}$$

$$\begin{aligned}
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x, |x-y|))} \frac{(x+|x-y|)^{2\lambda} r}{(x^{2\lambda} + |x-y|^{2\lambda})|x-y|} dm_\lambda(y) \right]^{1/2} \\
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0-y|))} \frac{r}{|x_0-y|} dm_\lambda(y) \right]^{1/2}.
\end{aligned}$$

This implies (3.3).

Now we estimate J_2 . By (3.4), we see that for $x_1 \in I$ and $r \geq \delta_i$, $i \in \mathbb{N}$, we have

$$\{y \in \mathbb{R}_+ \setminus 4I : \delta_i < |x-y| < \delta_i + 2r\} \subset \{y \in \mathbb{R}_+ \setminus 4I : |x-y| < 3r\} = \emptyset.$$

which implies that $J_2 = 0$. Therefore, in what follows, we assume $r < \delta_i$. Observe that

$$0 < m_\lambda(I(x_1, \delta_i + 2r)) - m_\lambda(I(x_1, \delta_i)) \lesssim (x_1 + \delta_i + 2r)^{2\lambda} r \lesssim (x_1 + \delta_i)^{2\lambda} r.$$

By this fact, Hölder's inequality, (2.1), (3.5), (3.4) and (3.2), we have

$$\begin{aligned}
J_2 &\leq [m_\lambda(I(x_1, \delta_i + 2r)) - m_\lambda(I(x_1, \delta_i))]^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}_+} \chi_{\{\delta_i < |x_1-y| < \delta_i + 2r\}}(y) \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) |R_{\Delta_\lambda}(x_1, y)|^2 |f_2(y)|^2 dm_\lambda(y) \right)^{1/2} \\
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{\delta_i < |x_1-y| < \delta_i + 2r\}}(y) \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2 (x_1 + \delta_i)^{2\lambda} r}{m_\lambda(I(x_1, |x_1-y|))^2} dm_\lambda(y) \right]^{1/2} \\
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_1, |x_1-y|))} \frac{(x_1 + |x_1-y|)^{2\lambda} r}{m_\lambda(I(x_1, |x_1-y|))} dm_\lambda(y) \right]^{1/2} \\
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_1, |x_1-y|))} \frac{(x_1 + |x_1-y|)^{2\lambda} r}{(x_1^{2\lambda} + |x_1-y|^{2\lambda})|x_1-y|} dm_\lambda(y) \right]^{1/2} \\
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0-y|))} \frac{r}{|x_0-y|} dm_\lambda(y) \right]^{1/2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_3 &\leq [m_\lambda(I(x_1, t_{i+1} + 2r)) - m_\lambda(I(x_1, t_{i+1}))]^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x_1-y| < t_{i+1} + 2r\}}(y) \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) |R_{\Delta_\lambda}(x_1, y)|^2 |f_2(y)|^2 dm_\lambda(y) \right)^{1/2} \\
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0-y|))} \frac{r}{|x_0-y|} dm_\lambda(y) \right]^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
J_4 &\leq [m_\lambda(I(x, \delta_i + 2r)) - m_\lambda(I(x, \delta_i))]^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}_+} \chi_{\{\delta_i < |x-y| < \delta_i + 2r\}}(y) \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) |R_{\Delta_\lambda}(x_1, y)|^2 |f_2(y)|^2 dm_\lambda(y) \right)^{1/2} \\
&\lesssim \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0-y|))} \frac{r}{|x_0-y|} dm_\lambda(y) \right]^{1/2}.
\end{aligned}$$

Now returning to the estimate of D_2 and combining the estimates of J_1, J_2, J_3 and J_4 , we can write

$$\begin{aligned} D_2 &\lesssim \left\| \left\{ \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0 - y|))} \frac{r}{|x_0 - y|} dm_\lambda(y) \right]^{1/2} \right\}_{\delta_i \in I_i, i \in \mathbb{N}} \right\|_{\mathcal{E}} \\ &\quad + \left\| \left\{ \left[\int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x_1-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0 - y|))} \frac{r}{|x_0 - y|} dm_\lambda(y) \right]^{1/2} \right\}_{\delta_i \in I_i, i \in \mathbb{N}} \right\|_{\mathcal{E}} \\ &=: D_{21} + D_{22}. \end{aligned}$$

A straight calculation involving (2.3) implies that

$$\begin{aligned} D_{21} &= \left[\sum_{i \in \mathbb{N}} \sup_{\delta_i \in I_i} \int_{\mathbb{R}_+} \chi_{\{t_{i+1} < |x-y| \leq \delta_i\}}(y) \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0 - y|))} \frac{r}{|x_0 - y|} dm_\lambda(y) \right]^{1/2} \\ &\lesssim \left[\int_{\mathbb{R}_+} \frac{|f_2(y)|^2}{m_\lambda(I(x_0, |x_0 - y|))} \frac{r}{|x_0 - y|} dm_\lambda(y) \right]^{1/2} \\ &\lesssim \left[\sum_{k=2}^{\infty} \int_{2^{k+1}I \setminus 2^k I} \frac{|f(y)|^2}{m_\lambda(I(x_0, |x_0 - y|))} \frac{r}{|x_0 - y|} dm_\lambda(y) \right]^{1/2} \\ &\lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} \left[\sum_{k=2}^{\infty} 2^{-k} \frac{m_\lambda(2^{k+1}I)}{m_\lambda(2^k I)} \right]^{\frac{1}{2}} \\ &\sim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}. \end{aligned}$$

Similarly, $D_{22} \lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}$. Consequently, we have

$$D_2 \lesssim \|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}.$$

Combining the estimate for D_1 , we complete the proof of Theorem 1.5.

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